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ATOMIC NUCLEUS AND HARMONIC MAPS

BY

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Abstract. There is a well-known case where the homographic action of a real 2×2 matrix is executed on two complex conjugated variables: the classical case of a Kepler motion. As a consequence of this, the space expanse of the atomic nucleus is physically represented by a specific harmonic map.

This is the essential fact that places the Skyrme theory as a logical continuation of the Newtonian natural philosophy.

Keywords: atomic nucleus; harmonic maps; Kepler motion; Skyrme theory.

1. The Classical Kepler Problem

The classical Kepler motion is the model depicting the revolution of planets around Sun, or the revolution of the electrons around nucleus, within the framework of classical dynamics. The image is usually rendered with reference

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to material points taken as pure positions, and can be dynamically explained with the help of Newtonian equations of motion. In vector notation these are:

$$\ddot{\vec{r}} + \frac{K}{r^2}\frac{\vec{r}}{r} = \vec{0} \tag{1}$$

Here K is a constant, $\vec{r}$ denotes the position vector, with respect to the center of force, of the material point whose motion is calculated. The constant K does not depend on quantities related to the point considered in motion, but only in cases where electric forces are involved. We can simplify the algebra leading to solution of Eq. (1) by restricting the geometry to the plane of motion, as a benefit of the vector expression of the centrality of force. If the generic coordinates of the point in motion are ξ and η say, the Eq. (1) is then equivalent to the system (Mittag and Stephen, 1992).

$$\ddot{\xi} + K\frac{\cos\phi}{r^2} = 0, \qquad \ddot{\eta} + K\frac{\sin\phi}{r^2} = 0 \tag{2}$$

with r and ϕ polar coordinates of the moving point, in the plane of motion, with respect to the attraction center. The magnitude of the rate of area swept by the position vector is then given by

$$\dot{a} \equiv \xi\dot{\eta} - \eta\dot{\xi} = r^2\dot{\phi} \tag{3}$$

By using K we can intehgrate the system (2), and thus obtain the analytical form of the trajectory. Let us firstly define the complex variable

$$z \equiv \xi + i\eta = re^{i\phi} \tag{4}$$

so that (2) can be written as

$$\ddot{z} + \frac{K}{r^2}e^{i\phi} = 0 \tag{5}$$

Now, by using (3) we eliminate r^2, so that

$$\ddot{z} + \frac{K}{\dot{a}}e^{i\phi}\dot{\phi} = 0 \quad \therefore \quad \dot{z} = i\left(\frac{K}{\dot{a}}e^{i\phi} + w\right) \tag{6}$$

where $w \equiv w_1 + iw_2$ is a complex constant of integration and has to be determined by *the initial conditions* of the problem. The analytical equation of motion can be extracted then directly, by calculating the area constant (3) with the help of the first result of integration given in Eq. (6). In polar coordinates the result is given by the following equation:

$$\frac{\dot{a}}{r} = \frac{K}{\dot{a}} + w_1\cos\phi + w_2\sin\phi \tag{7}$$

The shape of this trajectory can be found in Cartesian coordinates ξ and η, whereby we have, instead of (7) the second-degree curve – a conic:

$$\left(\frac{K^2}{\dot{a}^2}-w_1^2\right)\xi^2-2w_1w_2\xi\eta+\left(\frac{K^2}{\dot{a}^2}-w_2^2\right)\eta^2+2\dot{a}(w_1\xi+w_2\eta)=\dot{a}^2 \tag{8}$$

The center of this conic is *different than the center of the force, i.e.* the origin of the plane of motion in our case, but has the coordinates

$$\xi_0=-\frac{\dot{a}w_1}{\Delta},\quad \eta_0=-\frac{\dot{a}w_2}{\Delta};\quad \Delta\equiv\left(\frac{K}{\dot{a}}\right)^2-w_1^2-w_2^2 \tag{9}$$

In cases where $\Delta = 0$, the center of this trajectory is at infinity: the trajectory is a parabola.

If we assume that the center of the trajectory is at a finite distance with respect to the center of force, and referring the trajectory to this center by a translation: $x = \xi-\xi_0$, $y = \eta-\eta_0$, we can write

$$\left(\frac{K^2}{\dot{a}^2}-w_1^2\right)x^2-2w_1w_2xy+\left(\frac{K^2}{\dot{a}^2}-w_2^2\right)y^2=\frac{K^2}{\Delta} \tag{10}$$

The quadratic form from from (10) is completely characterized by the 2×2 special matrix

$$\mathbf{a}\equiv\begin{pmatrix}\dfrac{K^2}{\dot{a}^2}-w_1^2 & -w_1w_2\\ -w_1w_2 & \dfrac{K^2}{\dot{a}^2}-w_2^2\end{pmatrix} \tag{11}$$

The eigenvalues of this matrix are Δ and $K^2/\dot{a}^2$, with the corresponding eigenvectors

$$|e_1\rangle=\begin{pmatrix}\cos\omega\\ \sin\omega\end{pmatrix},\quad |e_2\rangle=\begin{pmatrix}-\sin\omega\\ \cos\omega\end{pmatrix};\quad w_1\equiv|w|\cos\omega,\quad w_2\equiv|w|\sin\omega \tag{12}$$

Therefore, the orientation of trajectory in its plane is defined by the initial conditions of the motion which, moreover, also properly define its relative asymmetry in the two directions of the axes. Specifically, the asymmetry is measured by the ratio of the two principal dimensions of the ellipse, which boils down to the fact that the magnitude $|w|$ is proportional with the eccentricity ‘e’ of trajectory. Indeed the semiaxes, customarily denoted by ‘a’ and ‘b’, and the eccentricity ‘e’ calculated with their values, are given by

$$a^2=\frac{K^2}{\Delta^2}, b^2=\frac{\dot{a}^2}{\Delta} \quad \therefore \quad e^2\equiv\frac{a^2-b^2}{a^2}=\left(\frac{\dot{a}}{K}\vec{w}\right)^2 \tag{13}$$

where $\vec{w}$ is the vector equivalent of the complex number ‘w’. This notation makes the vector nature of the eccentricity obvious. Therefore, the initial conditions can actually be expressed only in terms of some ‘contemporary’ magnitudes, to which we have only to add the partially controllable arbitrariness of an angle:

$$w_1=\frac{K}{\dot{a}}e\cdot\cos\omega, \quad w_2=\frac{K}{\dot{a}}e\cdot\sin\omega \tag{14}$$

This allows us, to a great extent, to *forget, at least partially, about the past*: as long as it is represented by the initial conditions, it can be found in contemporary measurable quantities. These are, obviously, the components of the vector $\vec{w}$ from Eq. (13). This observation has lead to the Newtonian explanation for the real planetary motions in terms of contemporary quantities. Indeed, a force is always contemporary, and the initial conditions of the motion – whatever they might have been in a remote past which is never within our reach – are then to be read, at least partially, in some contemporary parameters of motion, specifically the area constant and the eccentricity.

2. A Classical Description of the Nucleus

As it can be seen from Eqs. (13) the semiaxe ‘b’ can be imaginary for $\Delta < 0$, in which case the trajectories will be hyperbolic. Only if $\Delta > 0$, then the trajectories will be elliptic, so we can use them to describe the planetary motions. With respect to these facts, the parabolic trajectories are all characterized by points on the circle $\Delta = 0$, *i.e*.

$$\mu^2+\nu^2=1, \quad \mu\equiv e\cdot\cos\omega, \nu\equiv e\cdot\sin\omega \tag{15}$$

Therefore all possible finite Kepler motions *that a material point can have* around a center acting with a force inversely proportional to the square of distance can be attributed to the *whole interior of this circle*. This would mean that the motion of a planet or an electron indicates infinitely many possible initial conditions, from which it would have had to ‘choose’ so to speak. The actual motion of a planet is perceived as if having *unique initial conditions*. Any departure from this perception has always induced arguments about some actual perturbations acting on the planet. This could be partially true: the discovery of Neptune is an example. Let us turn to the origin of the problem, and direct our reasoning along the following lines: Kepler motion should have reality only as a ‘snapshot’, this fact cannot be denied, for it could not have been discovered otherwise. However the planetary motion, even if we confine its description to a Kepler motion, is a succession of such snapshots, which have to be brought together to make the whole picture. Foremost, we have to find the time scale of such a snapshot, and this is difficult. Fortunately, we have another possibility, opened by the above mentioned observations. Namely, there is an a priori metric geometry for the defining parameters of the snapshot, which are in fact the *initial conditions* of the dynamical problem describing it. This geometry defines in turn a kinematics, and that kinematics offers us a natural way to represent a real trajectory, by continuously connecting the snapshots in succession.

Indeed, it can be immediately seen that the above-mentioned freedom of the parameters defining the types of Kepler orbits, allows us to construct a Cayley-Klein geometry (Cayley, 1859; Klein, 1897) characterizing the variation of those orbits. We know that an absolute geometry is related to some conservation laws, at least as long as some realizations of SL(2, R) group structure are involved. And indeed, the absolute metric for the interior of the circle (15)

$$(ds)^2 = \frac{(1-\nu^2)(d\mu)^2 + 2\mu\nu(d\mu)(d\nu) + (1-\mu^2)(d\nu)^2}{(1-\mu^2-\nu^2)^2} \tag{16}$$

can be brought to the form of Poincaré metric

$$(ds)^2 = -4\frac{dh \cdot dh^*}{(h-h^*)^2} = \frac{(du)^2 + (dv)^2}{v^2} \tag{17}$$

by the following transformation of coordinates:

$$\mu = \frac{hh^*-1}{hh^*+1}, \nu = \frac{h+h^*}{hh^*+1} \quad \leftrightarrow \quad h \equiv u + iv = \frac{\nu + i\sqrt{1-\mu^2-\nu^2}}{1-\mu}; h^* = u - iv \tag{18}$$

The conservation laws for the metric (17) are represented by the following differential 1-forms

$$\omega_1 = \frac{du}{v^2}, \quad \omega_2 = 2\frac{udu + vdv}{v^2}, \quad \omega_3 = \frac{(u^2-v^2)du + 2uvdv}{v^2} \tag{19}$$

A natural description of the Kepler motion is the one in variables (e, ω), *i.e*. the eccentricity and the orientation of the orbit in its plane. In terms of these parameters the metric (16) becomes

$$(ds)^2 = \left(\frac{de}{1-e^2}\right)^2 + \frac{e^2}{1-e^2}(d\omega)^2 \tag{20}$$

We can rewrite this metric in a well-known form, by recalling that for elliptic trajectories ‘e’ is confined to the interval between –1 and +1, so that the change of parameter

$$e = \tanh\psi \tag{21}$$

is legitimate. With this the metric (20) becomes

$$(ds)^2 = (d\psi)^2 + \sinh^2\psi(d\omega)^2 \tag{22}$$

The complex parameter ‘h’ from Eq. (18) has now a direct connection with the theory of classical Newtonian potentials *via a harmonic map*. In order to show this relationship we write here ‘h’ from Eq. (18) in terms of (e, ω). We have:

$$h = i\frac{\cosh\chi + \sinh\chi \cdot e^{-i\omega}}{\cosh\chi - \sinh\chi \cdot e^{-i\omega}}, \qquad \chi \equiv \frac{\psi}{2} \tag{23}$$

As it happens, this equation represents a harmonic map from the usual space into the Lobachevsky plane having the metric (17), *provided* χ (and therefore ψ) *is a solution of the Laplace equation in free space*.

The stationary values of energy functional corresponding to the metric (17) describe the problem of harmonic correspondences between space and the hyperbolic plane. This is defined as the volume integral of an integrand obtained from that metric by transforming the differentials into space gradients (Eells and Sampson, 1964; Misner, 1978). The stationary values of energy functional therefore correspond to solutions of the Euler-Lagrange equations for a Lagrangian like

$$\Lambda \equiv -4\frac{\nabla h \cdot \nabla h^*}{(h - h^*)^2} \tag{24}$$

These are

$$(h - h^*) \cdot \nabla^2 h - 2(\nabla h)^2 = 0 \tag{25}$$

and its complex conjugate. Then it is easy to see, by a direct calculation, that h from (23) verifies this equation when χ is a solution of Laplace equation, and ω is arbitrary, in the sense that it does not depend on the position in space. Nevertheless, it might depend on the local time of the Newtonian dynamics, as it turns out to be the case in the particular problems related to the case of the damped harmonic oscillator (Mazilu, 2004). We shall discuss a particular aspect of such dynamical problem later in this work.

3. Casting New Light on the Old Nucleus

At the confluence of geometry and dynamics, physics has found that the planetary motion in the case of solar system, as well as the electronic motion in the case of classical atomic model, can be fairly depicted as a Kepler problem. This condition, let us recall once again, is rigorously satisfied only in cases where the bodies are point-like. The vector representing eccentricity in a Kepler orbit is related to the initial velocity that a planet, or an electron in the case of atom, is assumed to have had when it 'started' orbiting around Sun, respectively around the atomic nucleus. One can rightfully say that by means of the *present measurements* of the eccentricities, we have at least a 'partial' access to the past of the planets or electrons, as the case may occur.

One of the most important conclusions of this perception is that the indecision in the initial conditions of the Kepler motion describing the planetary or atomic dynamics, has a precise geometrical form: it is the geometry of the interior of the unit circle, therefore the hyperbolic geometry of Lobachevsky. As in the case of planets these eccentricities are small, in real space terms the hyperbolic geometry applies only to a small space region, practically concentrated to the volume of the Sun itself. The same can be said, by analogy, of the nucleus – if the atom is discussed. It is therefore conceivable that this hyperbolic geometry would describe the content, and the manner of producing, of both the solar and nuclear energy, by a kind of deformation process of a continuous structure of their matter. This is actually what Lord Kelvin tried to explain by the middle of the 19^{th} century, regarding the source of solar energy.

Here the most important conclusion though, is that this method can be thought of as really ascribing 'a space expanse' of the nuclear matter, in the form of *harmonic surfaces in space*, to the regions of space spreading over ranges whose measure is the eccentricity of the Kepler motion. In other words, the harmonic maps from the physical space to Lobachevsky plane are intimately related either to the physics of Sun, in the case of planetary system, or to the physics of nucleus in the case of classical atomic model. We could not find any signs, in physics today, of some such theoretical description in the first case, *i.e*. the case of Sun. That is, if we don't consider the contemporary preoccupations with the seismicity of Sun and planets, which, from some points of view, may qualify as an application of harmonic mappings. However, such a conclusion seems to be fair in the case of atomic model, where the nuclear matter can be described by harmonic maps, and more often at that lately, by *the model of Skyrme*.

Before getting into that subject, let us make nevertheless a connection with the conclusions from Mazilu and Agop (2012), regarding the space expanse of the atomic nucleus. In the description above, the central expanse of the atomic space plainly qualifies as the space into which the electron plunges, or from which it is ejected, in some classical 'Wilson processes' that involve or not a production of light. In other words, if the condition of material point of the nucleus is the one making us reluctant to accept the reality of such processes, then it is time to forget about it. Indeed, the contemporary space expanse of the atomic nucleus, like the contemporary space expanse of the Sun itself for that matter, should be the reflection into present of a distant past, as it appears in the description of the planetary system by Kepler motions.

This fact puts an interesting spin – properly, as well as figuratively, as we will see here – on the ejection of an electron or its absorption by the nucleus. When plunging into the nucleus, an electron appears as if it goes into the past, while when it is ejected from the nucleus it suddenly pops up into the present. The past and the present of an electron meet therefore inside the nuclear matter. For once, part of this process – popping up into present – has been met in a popular model involving the macroscopic counterpart of the atom – the planetary system. Indeed, the most familiar cosmogonic model of the planetary system is that in which the planets are born from a primary nebula by a process of ejection due to pure mechanical effect of the rotation. By carrying the analogy in reverse – *i.e*. from the atomic model to the planetary one – one might say that the cosmogonic model misses an essential part, corresponding to the ingestion of the electron by nucleus. Indeed, it is momentarily hard to say, within astrophysical experience today, what would be a cosmogonic process that corresponds to this quite natural one from the atomic realm.

In hindsight, this transition present-past and vice versa means actually to offer a physical content to time. The fact was brought to light for the first time by Richard Feynman in the construction of his famous graphs (Feynman, 1948; Feynman, 1949). These represent the transition from the usual continuous description in space and time by differential equations, to the aparent accidentality of events in the atomic realm. They make obvious the necessity of discussing the time by its physical content. The positron, for instance, is an electron going back in time: while for the electron the time goes normally, toward future, for the positron the time goes toward past.

One other pertinent observation here is related to the metric (22). It is well-known in theoretical physics: the Fock metric of the velocity space in special relativity (Fock, 1964). Indeed, this metric can be produced as an absolute metric in the three-dimensional space of velocities, in view of the fact that all possible velocities of the matter formations in the universe are smaller than the speed of light. The resulting metric is given by Eq. (20) where e is this time the ratio of the velocity in matter and the speed of light, and ω is the metric on the unit sphere. One can say that the only thing well specified here is the speed of light. All the other velocities, regarding material bodies are contingencies. This is why the Fock metric is well suited in producing a priori minimum information measures in velocity space (Evrard, 1995). The metrics (20) or (22) provide just such an a priori minimal measure, based on the same principles as the relativistic ones. As a matter of fact, their very source, the absolute metric (16) involves the special metric tensor, characteristic to a constant curvature space. This conveys properties of universality to a special geometrization of the harmonic maps representing the nucleus.

4. Conclusions

The main conclusions of the present paper are the following:

i) The classical Kepler problem in correlation with the Lobachevsky plane properties is established.

ii) Since the above correlations implies the functionality of harmonic maps by means of an operational procedure, an alternative to the Skyrme model for the nucleus can be identified.

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NUCLEU ATOMIC ȘI MAPE ARMONICE

(Rezumat)

Este cunoscut faptul că acțiunea omografică a matricilor reale 2x2 devine executorie prin intermediul a două variabile complexe: cazul clasic al unei mișcări de tip Kepler. În consecință, expansiunea spațială a nucleului atomic poate fi mimată prin intermediul unei mape armonice. Într-un asemenea context, modelul lui Skyrme poate fi admis ca o extensie naturală a filosofiei newtoniene.